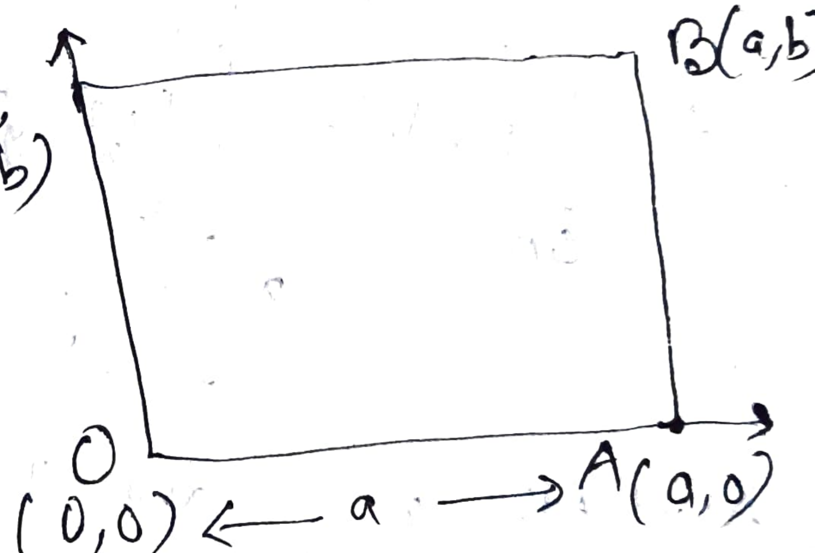
Vector Integration

I: Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where

$\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ and curve C is the rectangle in the xy plane bounded by $y=0,\ x=a,\ y=b,\ x=0$.

Soln



In xy-plane,
$z = 0 \Rightarrow dz = 0.$

$\because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\Rightarrow \vec{r} = x\vec{i} + y\vec{j}$

$\Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j}$

$\because \vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$

$\Rightarrow \vec{F} \cdot d\vec{r} = \left[(x^2+y^2)\vec{i} - 2xy\vec{j}\right] \cdot (dx\vec{i} + dy\vec{j})$

$= (x^2+y^2)dx - 2xy\,dy.$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r}$$

$$+ \int_{CO} \vec{F}\cdot d\vec{r} \quad \text{———} (1)$$

Along OA, $x = 0$ to a and $y = 0$.
$\Rightarrow dy = 0$ and $x = 0$ to a.

$$\therefore \int_{OA} \vec{F}\cdot d\vec{r} = \int_{x=0}^{a} \left[(x^2+y^2)\,dx - 2xy\,dy\right] = \int_0^a x^2\,dx$$

$$= \frac{a^3}{3}.$$

Along AB $x = a$ and $y = 0$ to b
$\Rightarrow dx = 0$ and $y = 0$ to b.

$$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int_{y=0}^{b} \left[(x^2+y^2)\,dx - 2xy\,dy\right] = \int_0^b -2ay\,dy$$

$$= -ab^2.$$

Along BC

$y = b$ and $x = a$ to 0

$\Rightarrow x = a$ to 0 and $dy = 0$.

$$\therefore \int_{BC} \vec{F} \cdot d\vec{r} = \int_{x=a}^{0} (x^2 + b^2)\, dx = \left[\frac{x^3}{3} + b^2 x\right]_a^0$$

$$= \frac{1}{3}(0 - a^3) + b^2(0 - a)$$

$$= -\frac{a^3}{3} - ab^2.$$

Along CO

$x = 0$ and $y = b$ to 0

$\Rightarrow y = b$ to 0 and $dx = 0$.

$$\int_{CO} \vec{F} \cdot d\vec{r} = \int_{y=b}^{0} 0 = 0$$

Hence, from eq (1), we've

$$\int_{C} \vec{F} \cdot d\vec{r} = \cancel{\frac{a^3}{3}} - ab^2 - \cancel{\frac{a^3}{3}} - ab^2$$

$$= -2ab^2$$

Q. If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$, find $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve in xy-plane $y = 2x^2$ from (0, 0) to (1, 2).

Soln In xy-plane, $z = 0 \Rightarrow dz = 0$

$\therefore \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = x\vec{i} + y\vec{j}$

$\Rightarrow d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$\therefore \vec{F}\cdot d\vec{r} = 3xy\,dx - y^2\,dy.$

Given, $y = 2x^2 \Rightarrow dy = 4x\,dx$

$\Rightarrow \vec{F}\cdot d\vec{r} = 6x^3\,dx - 16x^5\,dx$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} (6x^3\,dx - 16x^5\,dx)$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{3}{2} - \frac{8}{3}$$

$$= \frac{9-16}{6} = -\frac{7}{6}$$